Geodesic theory of Lagrangian Coherent Structures

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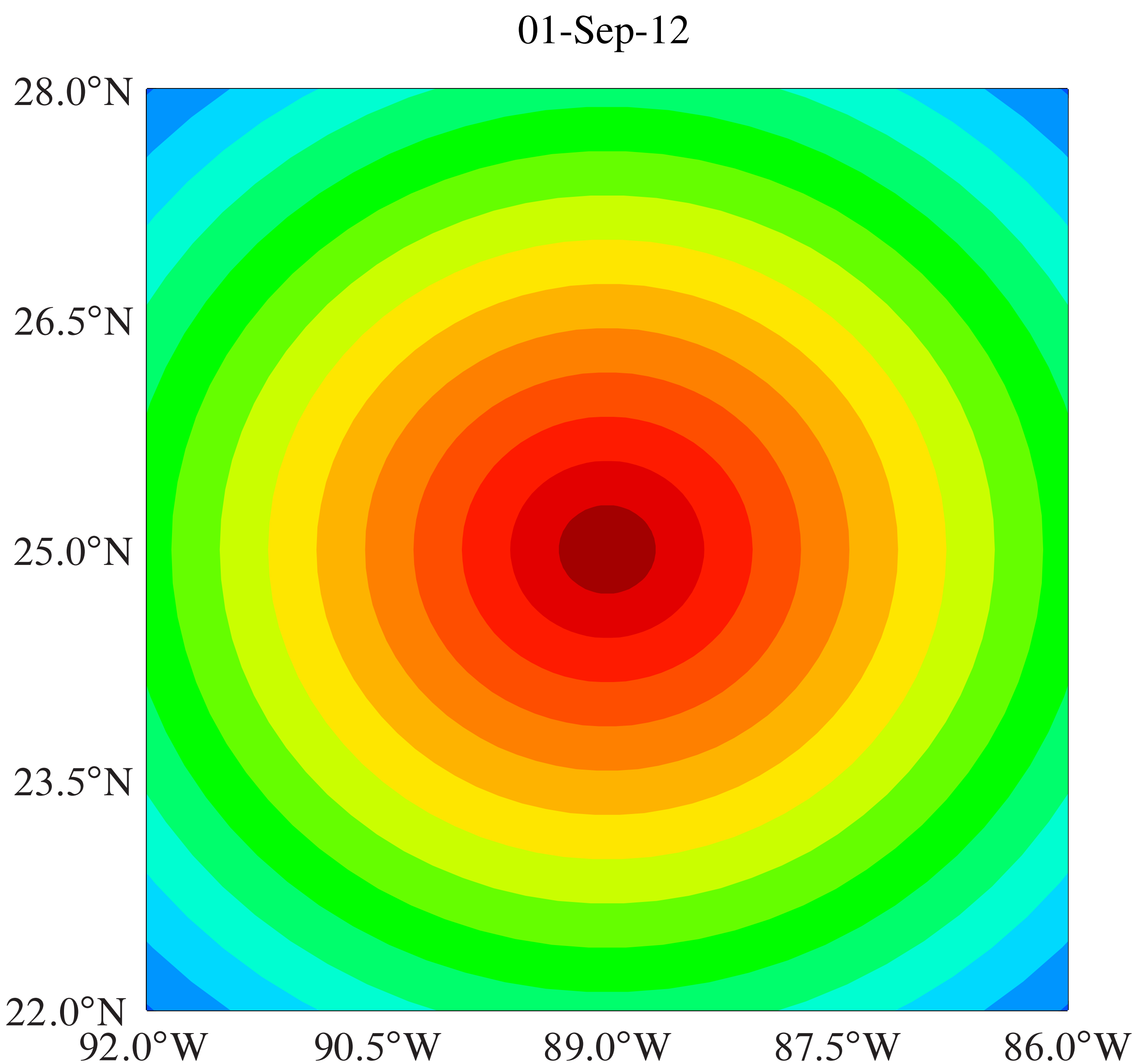
Motivation



Suggests organizing ***Lagrangian Coherent Structure (LCS)***.

Main result

Elliptic LCS (eddy boundaries), ***hyperbolic LCS*** (invariant-manifold-like) and ***parabolic LCS*** (shear jet cores) as (null-)geodesics of appropriately defined (Lorentzian) metrics.

References

Farazmand et al. (2013). *Physica D*, submitted (arXiv:1308.6136).

MJO et al. (2013). *GRL*, submitted.

Farazmand & GH(2013). Preprint.

Farazmand & GH (2013). *Chaos* 23, 023101.

GH & FJBV (2013). *JFM*, in press (arXiv:1308.2352).

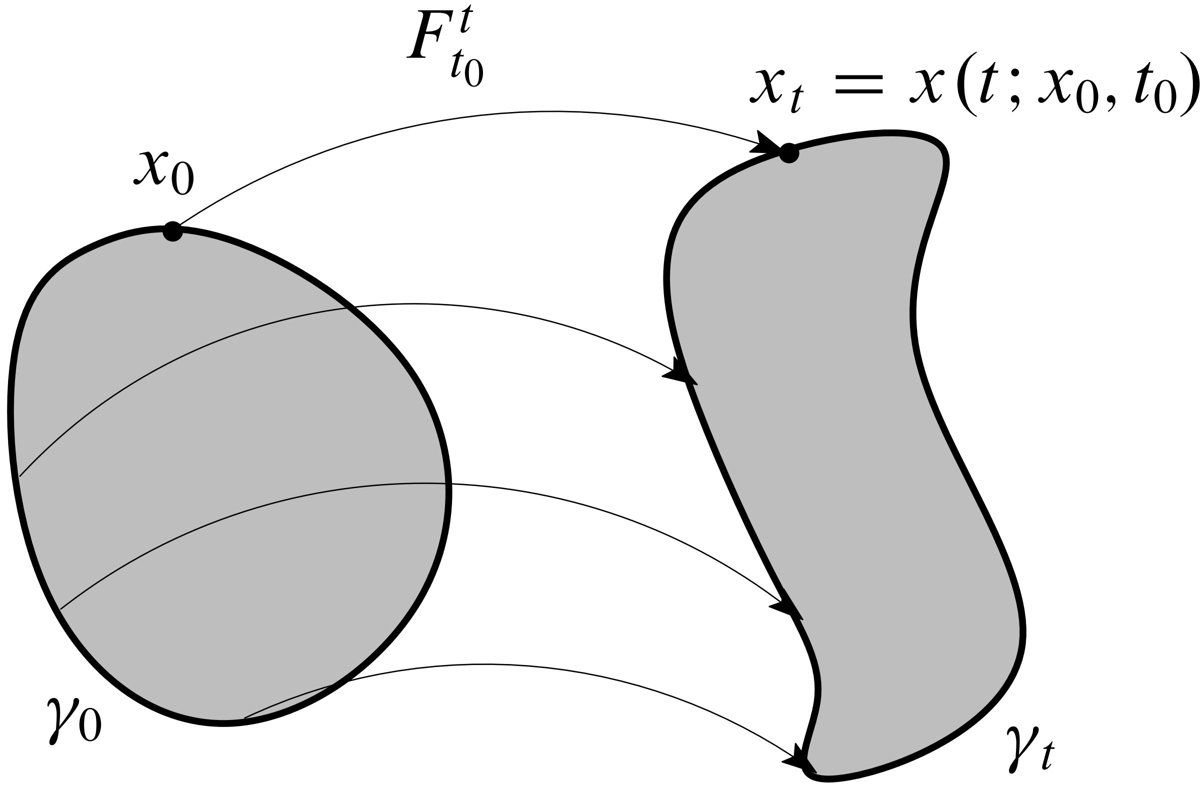
FJBV et al. (2013*a*). *JPO*, in press.

MJO & GH (2012). *PNAS* 109, 4738.

GH & FJBV (2012). *Physica D* 241, 168.

Mathematical setup

$$\dot{x} = v(x,t), \quad x \in U \subset \mathbb{R}^2, \quad t \in [t_0, t_1] \subset \mathbb{R}$$



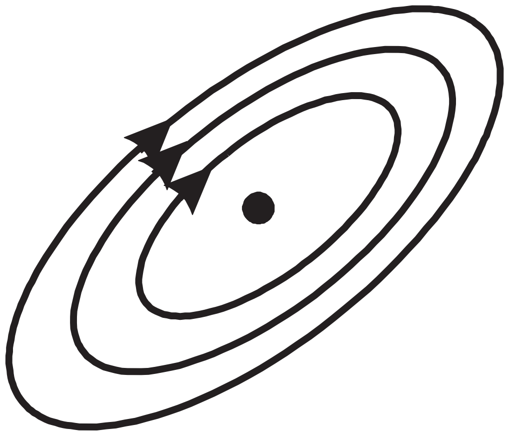
$$C_{t_0}^t(x_0) = \mathrm{D}F_{t_0}^t(x_0)^\top \mathrm{D}F_{t_0}^t(x_0)$$

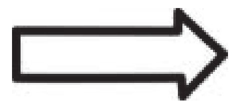
$$C_{t_0}^t \xi_i = \lambda_i \xi_i, \quad 0 < \lambda_1 \le \lambda_2, \quad \langle \xi_i, \xi_j \rangle = \delta_{ij}$$

$$x \mapsto Q(t)x + b(t) : C_{t_0}^t \mapsto C_{t_0}^t$$

Why objectivity (frame invariance) matters

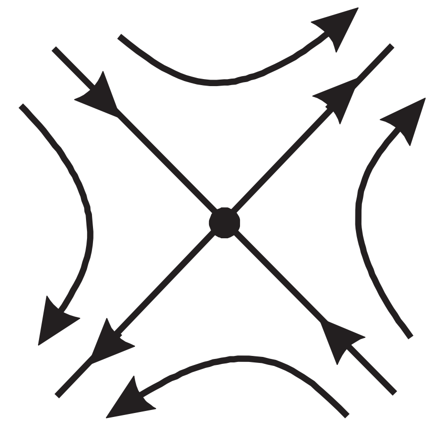
$$v(x,t) = \begin{pmatrix} \sin 4t & 2+\cos 4t \\ -2+\cos 4t & -\sin 4t \end{pmatrix} x \qquad \tilde{v}(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{x}$$

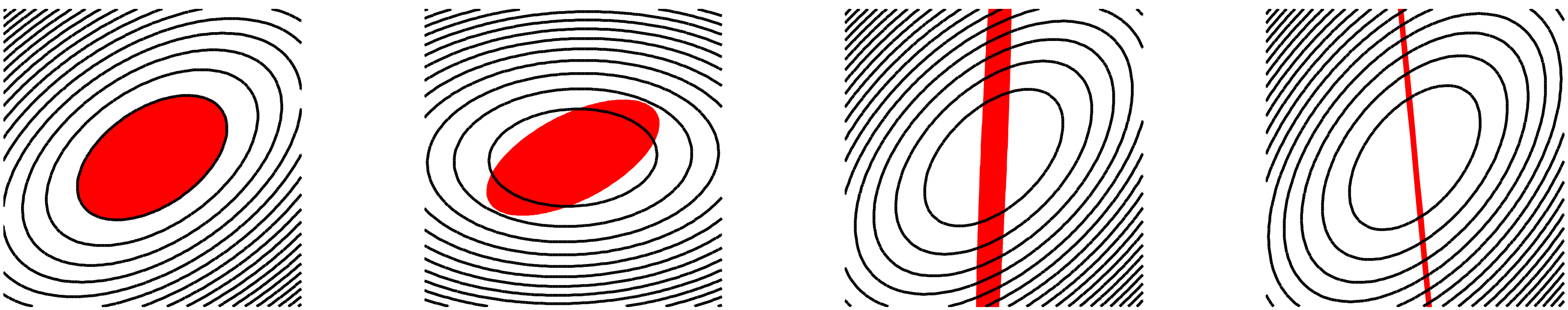




Switch to rotating frame:

$$x = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \tilde{x}$$

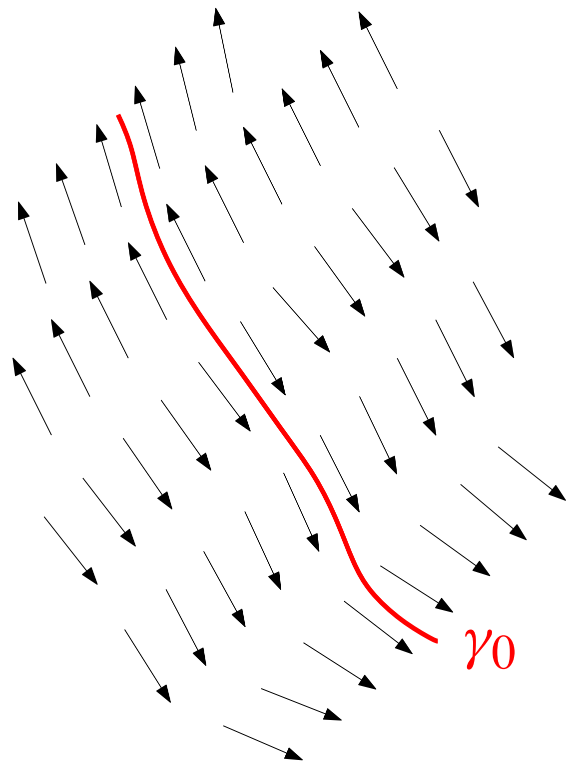




Truly unsteady flows have no distinguished frame—remain unsteady in any frame (Lugt, 1979). Conclusions about flow structures should not dependent on frame chosen.

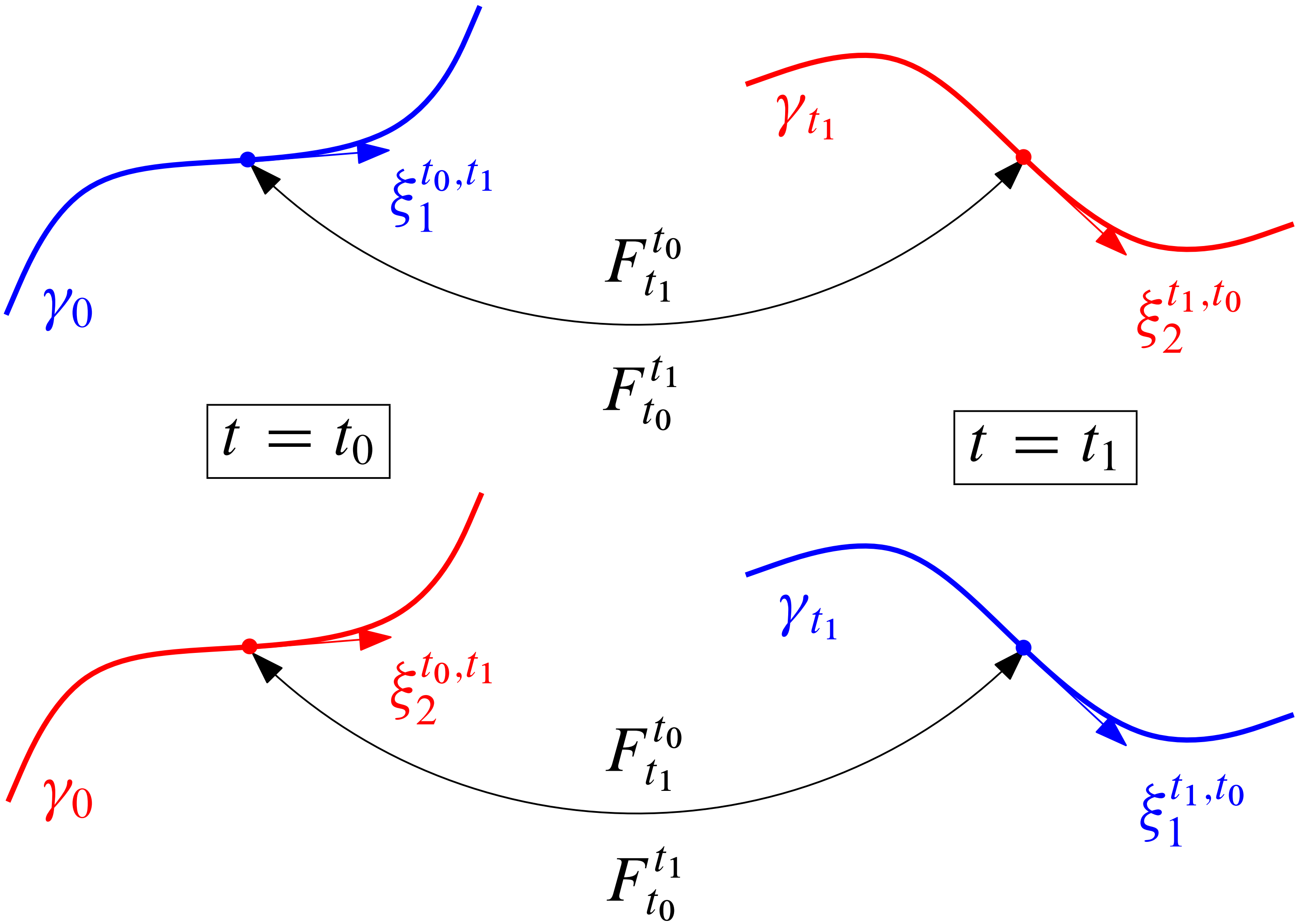
Strainlines, stretchlines, and λ-lines

$$\mathbb{R}^+ \supset [s_1, s_2] \ni s \mapsto r(s) \in \gamma_0$$



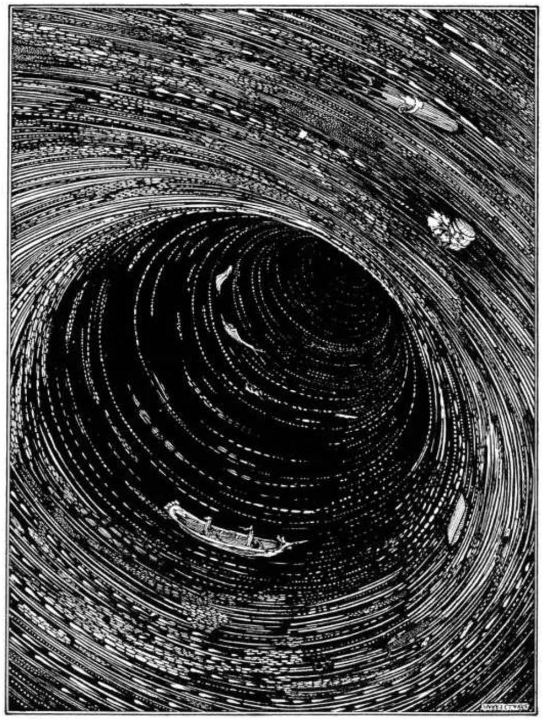
$$r' = \xi_1 \quad \text{or} \quad \xi_2 \quad \text{or} \quad \eta_\lambda^\pm := \sqrt{\frac{\lambda_2 - \lambda^2}{\lambda_2 - \lambda_1}}\xi_1 \pm \sqrt{\frac{\lambda^2 - \lambda_1}{\lambda_2 - \lambda_1}}\xi_2$$

From strainlines to stretchlines and vice versa

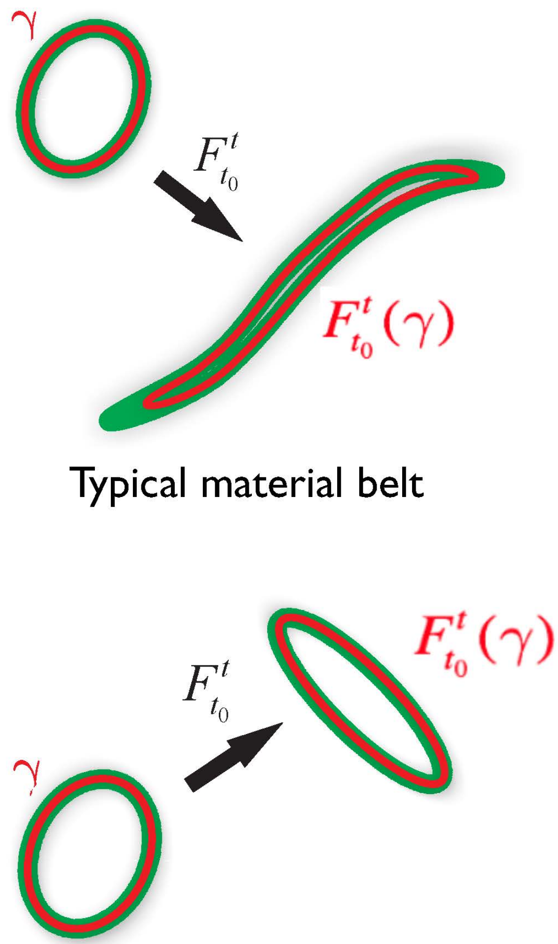


(Farazmand & Haller, 2013)

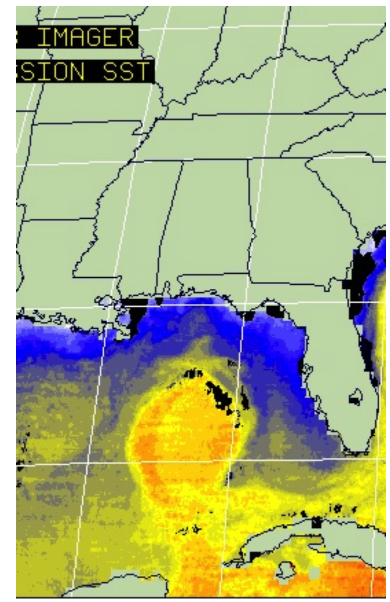
Elliptic LCS (Haller & FJBV, 2013)



Harry Clarke's illustration for "A Descent into a *Maelström*" by Edgar Allan Poe (1841)



Coherent material belt



"The edge of the whirl was represented by a broad belt of gleaming spray; but no particle of this slipped into the mouth of the terrific funnel…"

Variational principle

$$\mathbb{T} \ni s \mapsto r(s) \in \gamma_0 : \text{loop}$$

$$l_{t_0}(r') := \sqrt{\langle r', r' \rangle}, \quad l_t(r, r') := \sqrt{\langle r', C_{t_0}^t(r) r' \rangle}$$

$$Q(\gamma_0) := \oint \frac{l_t(r, r')}{l_{t_0}(r')} \, \mathrm{d}s$$

$$s \mapsto r(s) + \varepsilon h(s) \in \gamma_0^\varepsilon, \quad Q(\gamma_0^\varepsilon) = Q(\gamma_0) + O(\varepsilon^2) \iff \delta Q(\gamma_0) = 0$$

$$l_t^2(r, r') / l_{t_0}^2(r') = \lambda^2 = \text{const}$$

$$\delta \mathcal{E}_\lambda(\gamma_0) = 0, \quad \mathcal{E}_\lambda(\gamma_0) := \oint g_\lambda(r)(r', r') \, \mathrm{d}s$$

$$g_\lambda(x_0)(u, u) := \langle u, E_\lambda(x_0) u \rangle$$

$$E_\lambda(x_0) := \tfrac{1}{2}(C_{t_0}^t(x_0) - \lambda^2 \, \mathrm{Id})$$

Coherent Lagrangian eddy boundaries

- Solutions, called ***λ-loops***, satisfy implicit ODE:

$$l_t^2(r, r') - \lambda^2 l_{t_0}^2(r') = \langle r', E_\lambda(r) r' \rangle \ = 0.$$

- Stretch by same factor λ from t_0 to t—coherent cores of coherent material belts.
- We seek λ-loops tangent to linear combinations of eigenvectors of $C_{t_0}^t$; leads to explicit ODE:

$$r' = \eta_\lambda^\pm(r),$$

$$\eta_\lambda^\pm = \sqrt{\frac{\lambda_2 - \lambda^2}{\lambda_2 - \lambda_1}}\,\xi_1 \pm \sqrt{\frac{\lambda^2 - \lambda_1}{\lambda_2 - \lambda_1}}\,\xi_2.$$

- $\eta_\lambda^\pm$ constitute rotated vector field—λ-loops are nonintersecting (Duff, 1953).
- Observable boundary given by outermost of concentric λ-loops.

Cosmological analogy

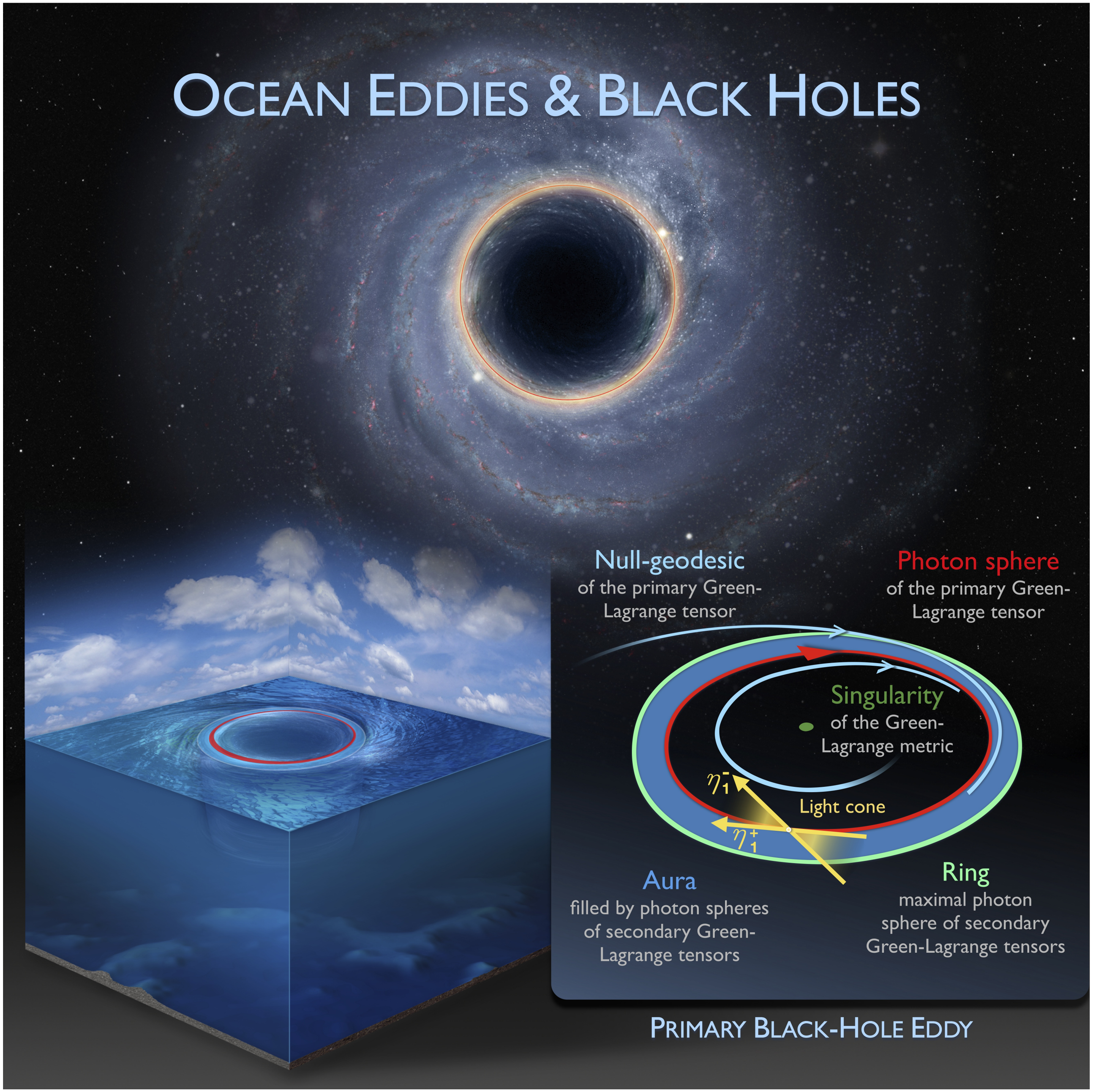
- A each $x_0 \in U_\lambda$ the quadratic form

$$g_\lambda(x_0)(u,u) = \langle u, E_\lambda(x_0)u\rangle$$

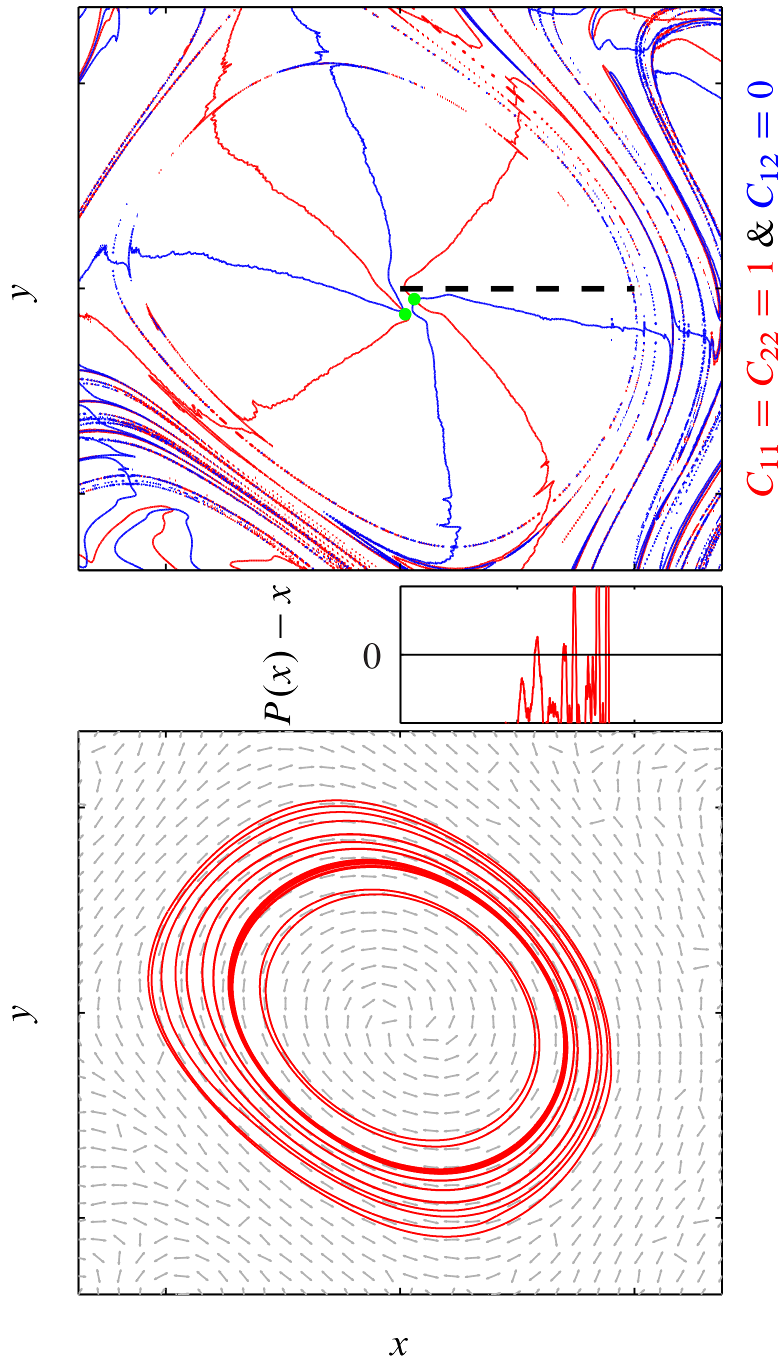
  defines a Lorentzian metric. Then (U_λ, g_λ) can be view as Lorentzian 2-manifold—a relativistic spacetime.
- Light travels along curves where a Lorentzian metric vanishes—null-geodesics.
- Closed null-geodesics (photon spheres) enclose black holes.
- Because λ-loops are closed geodesics of $g_\lambda(r)(r', r') = 0$:
  - ▷ outermost $\lambda = 1$ loop: ***primary black-hole eddy boundary***—super coherent
  - ▷ outermost $\lambda \neq 1$ loop: ***secondary black-hole eddy boundary***—coherent
- In the Hamiltonian case ($\operatorname{div} v = 0$), primary BH eddies preserve area—creates further coherence; most closely related to KAM tori.

Detection of black-hole eddies

- Strain eigenvectors ξ_1 and ξ_2 become ill-defined at points where $\lambda_1 = \lambda_2 = 1$—singularities of E_1.
- Null-geodesics of g_λ cannot be extended to such points—$\eta_\lambda^\pm$ are ill-defined there.
- Black holes are believed to contain Penrose–Hawking singularities—analogous to singularities of E_1.
- It can be proved (Beem et al., 1999) that close null-geodesics of g_λ (i.e., λ-loops) must contain singularities of E_1.
- We exploit this property to detect BH eddy candidate regions.

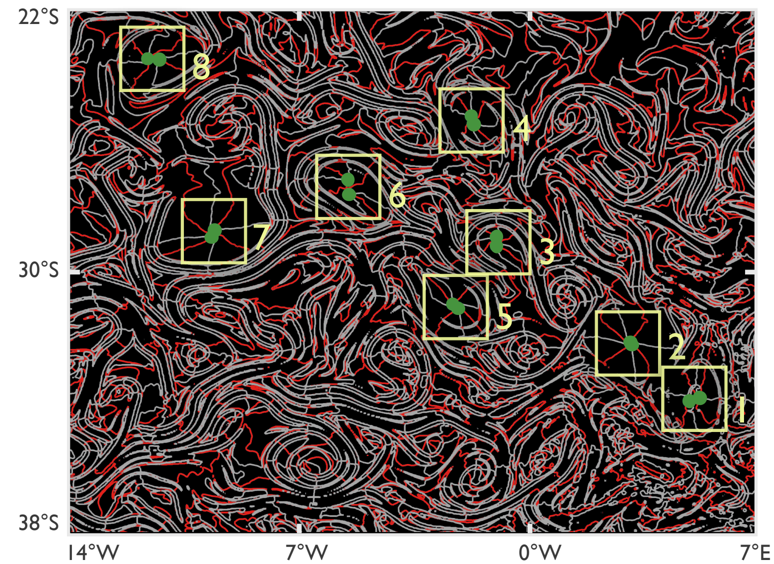

OCEAN EDDIES & BLACK HOLES
Null-geodesic
of the primary Green-Lagrange tensor
Photon sphere
of the primary Green-Lagrange tensor
Singularity
of the Green-Lagrange metric
η_1^-
Light cone
η_1^+
Aura
filled by photon spheres of secondary Green-Lagrange tensors
Ring
maximal photon sphere of secondary Green-Lagrange tensors
PRIMARY BLACK-HOLE EDDY

Computation of λ-loops

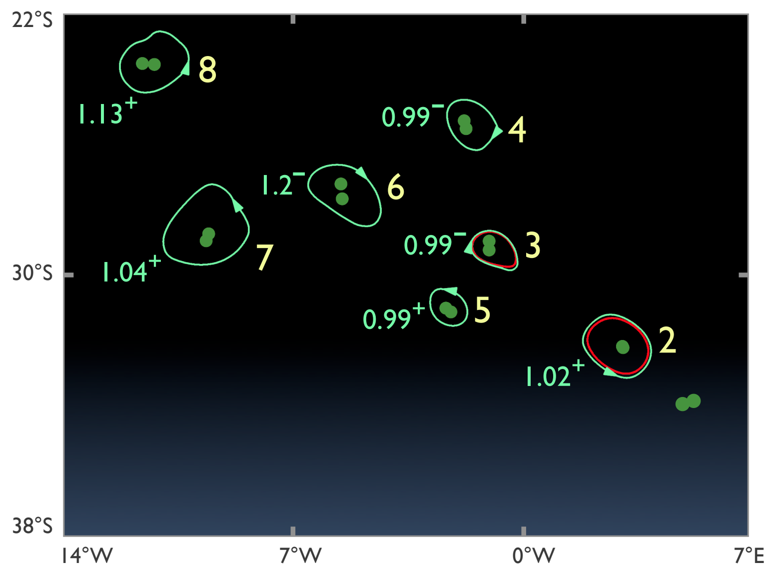


BH eddies in the South Atlantic

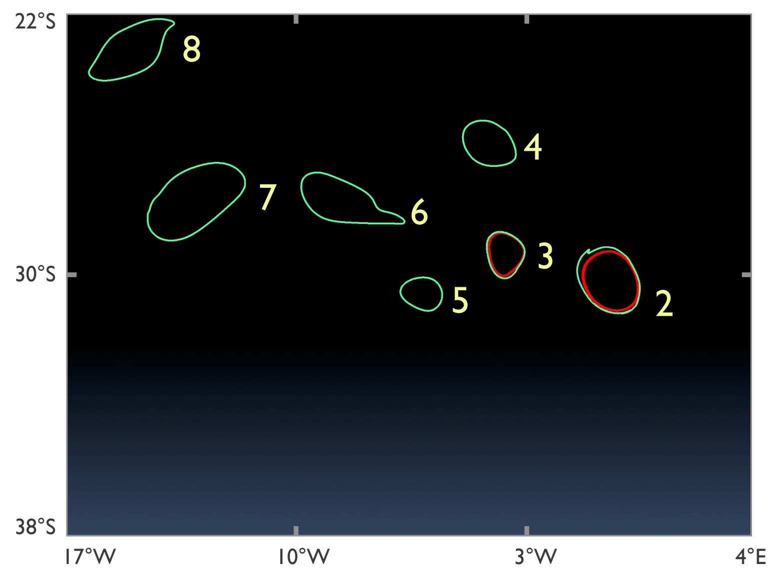
Metric singularities & black-hole eddy candidates on 24-Nov-06



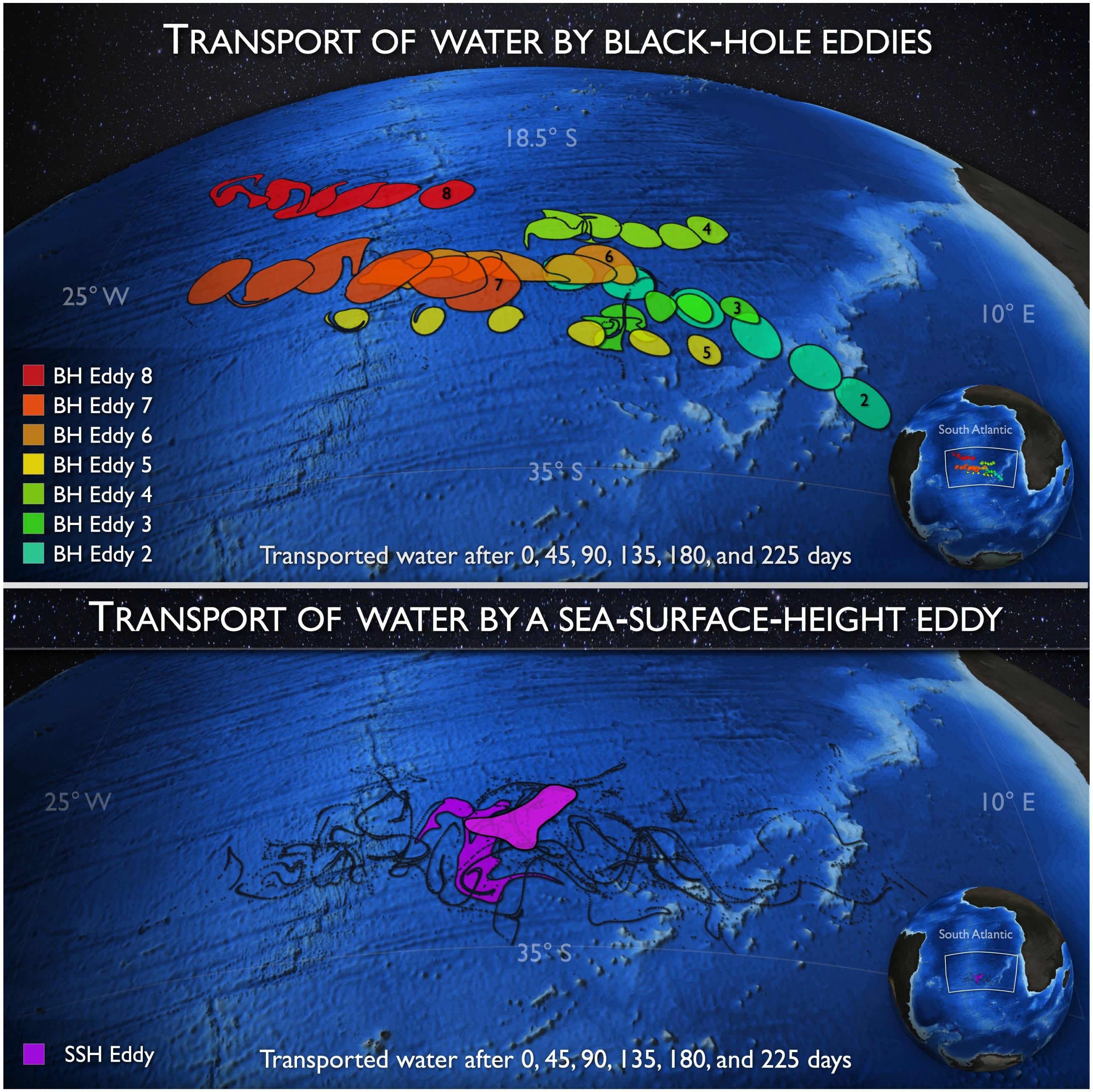
Black-hole eddy boundaries on 24-Nov-06



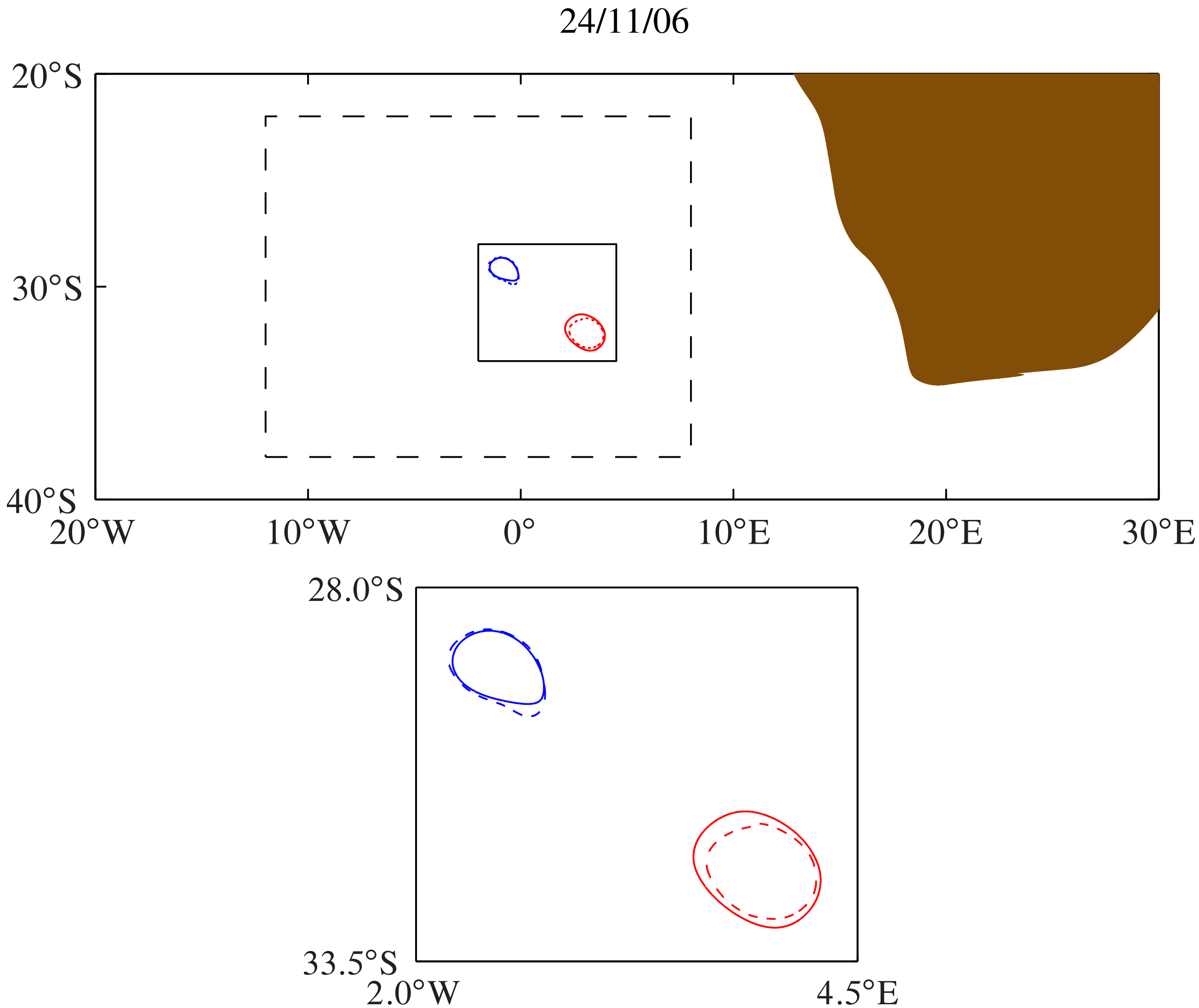
Black-hole eddy boundaries on 22-Feb-07



Long-term advection of BH and SSH eddies

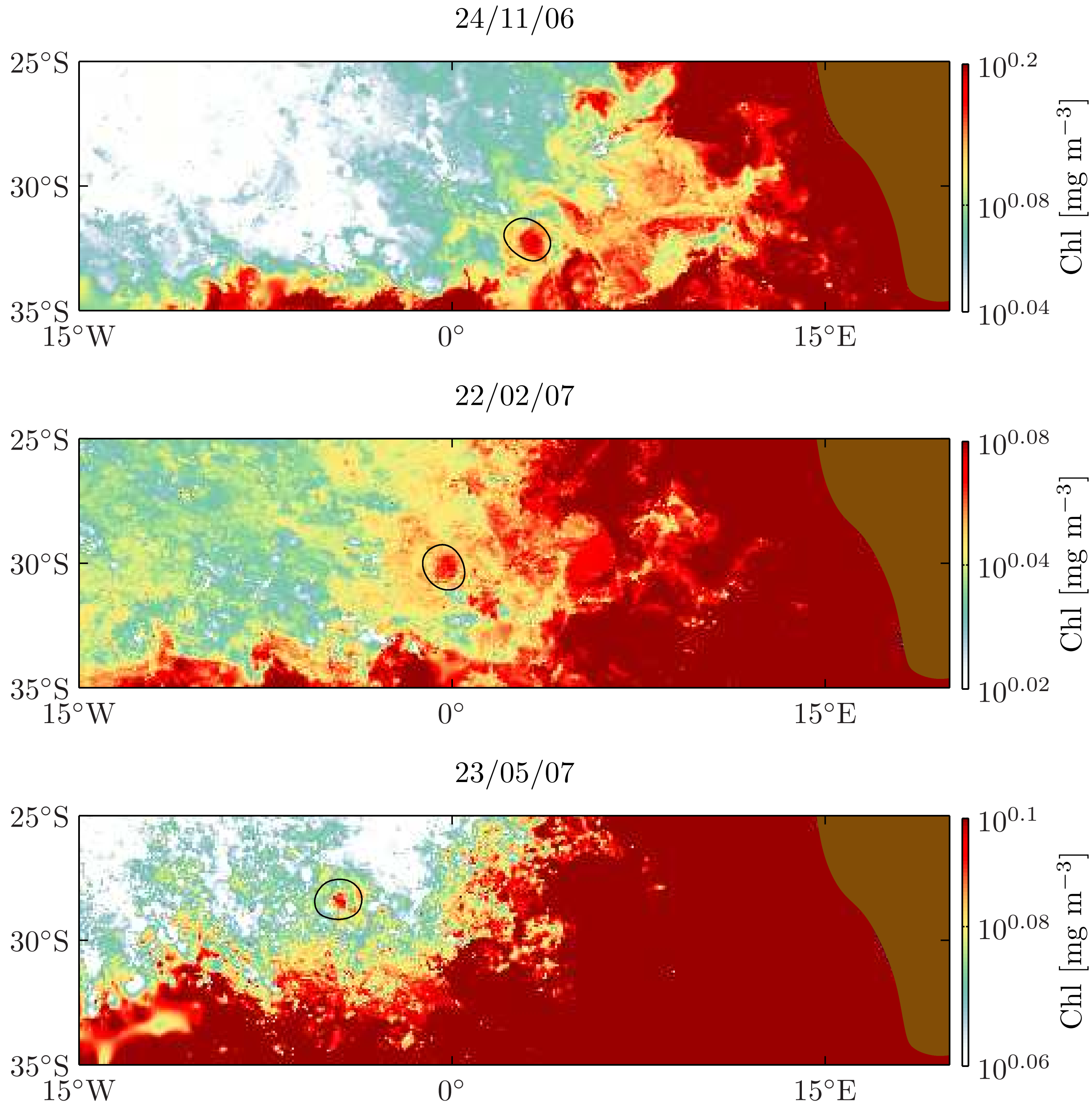


Robustness under velocity degradation

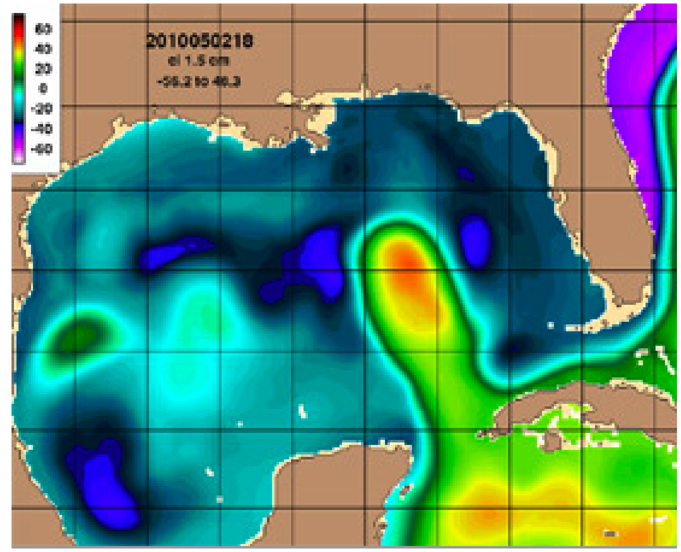


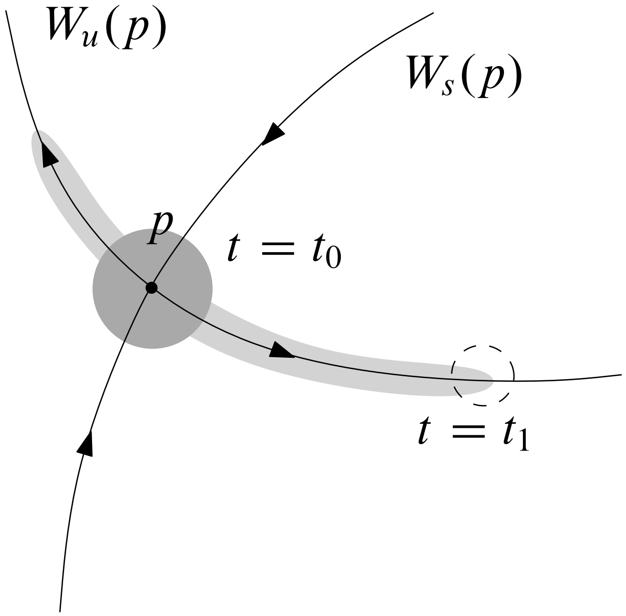
Direct consequence of structural stability of limit cycles.

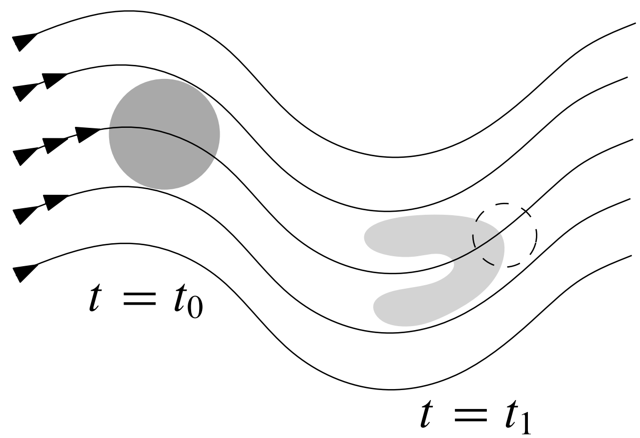
Reality check: surface ocean chlorophyll



Hyperbolic and parabolic LCS (Farazmand et al. 2013)

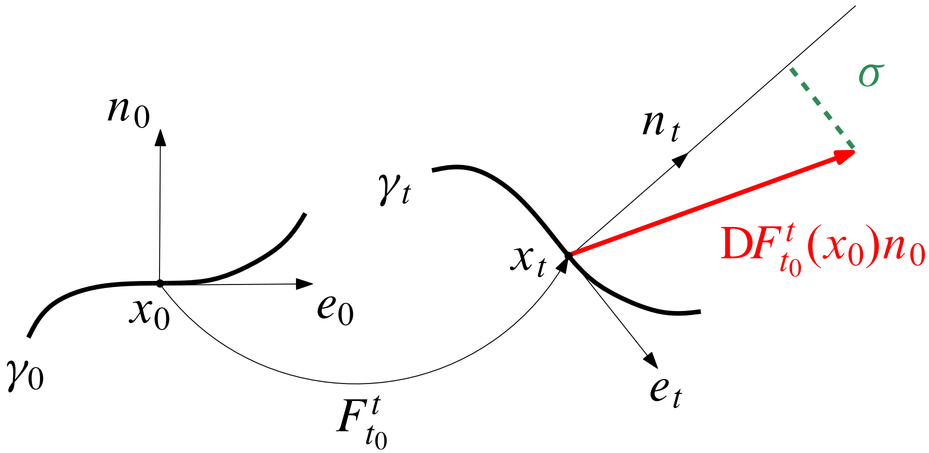






Shear vanishes locally.

Variational principle (Farazmand et al. 2013)



$$\sigma(r,r') = \frac{\langle r', D(r) r'\rangle}{\sqrt{\langle r', C_{t_0}^t(r) r'\rangle \langle r', r'\rangle}}$$

$$D := \frac{1}{2}(C_{t_0}^t \Omega - \Omega C_{t_0}^t)$$

$$\Sigma(\gamma_0) := \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \sigma(r, r')\, \mathrm{d}s$$

$$\Sigma(\gamma_0^\varepsilon) = \Sigma(\gamma_0) + O(\varepsilon^2) \iff \delta\Sigma(\gamma_0) = 0$$

$$\delta\Sigma(\gamma_0) = \langle \partial_{r'}\sigma, h\rangle|_{s_1}^{s_2} + \int_{s_1}^{s_2} \left(\partial_r \sigma - \frac{\mathrm{d}}{\mathrm{d}s} \partial_{r'}\sigma \right) h\, \mathrm{d}s$$

$$\text{free} : C_{t_0}^{t_1}(r(s_1)) = C_{t_0}^{t_1}(r(s_2)) = \mathrm{Id}$$

$$\text{fixed} : h(s_1) = h(s_2) = 0$$

Variational principle (Farazmand et al. 2013)

$$\frac{\langle r', D(r) r' \rangle}{\sqrt{\langle r', C_{t_0}^{t}(r) r' \rangle \langle r', r' \rangle}} = \mu = \text{const}$$

$$\mu = 0 : \langle r', D(r) r' \rangle = 0 \iff r' \| \xi_i$$

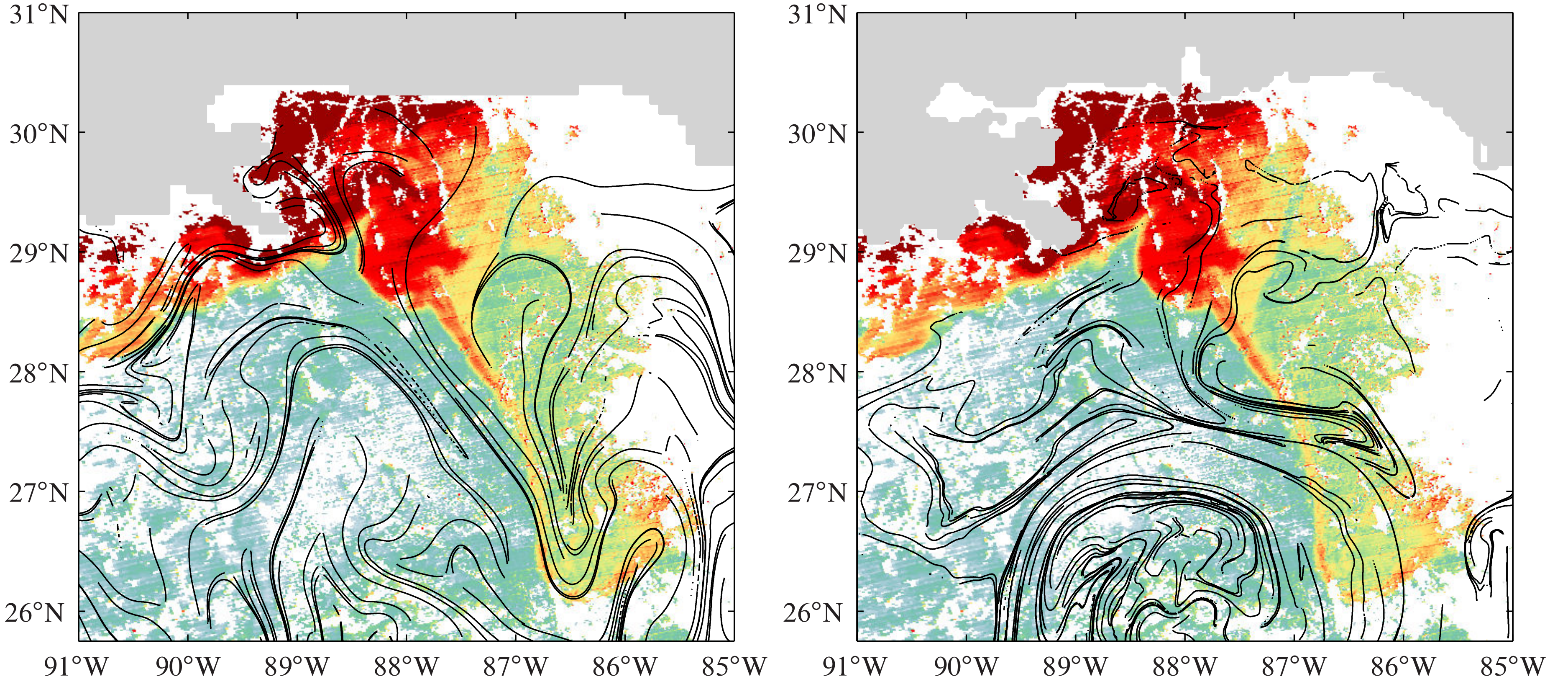
parabolic LCS : strain- and stretchlines connecting CG singularities
hyperbolic LCS : strain- or stretchlines

$$\delta \mathcal{E}(\gamma_0) = 0, \quad \mathcal{E}(\gamma_0) := \int_{s_1}^{s_2} g(r)(r', r')\, \mathrm{d}s$$

$$g(x_0)(u, u) := \langle u, D(x_0) u \rangle$$

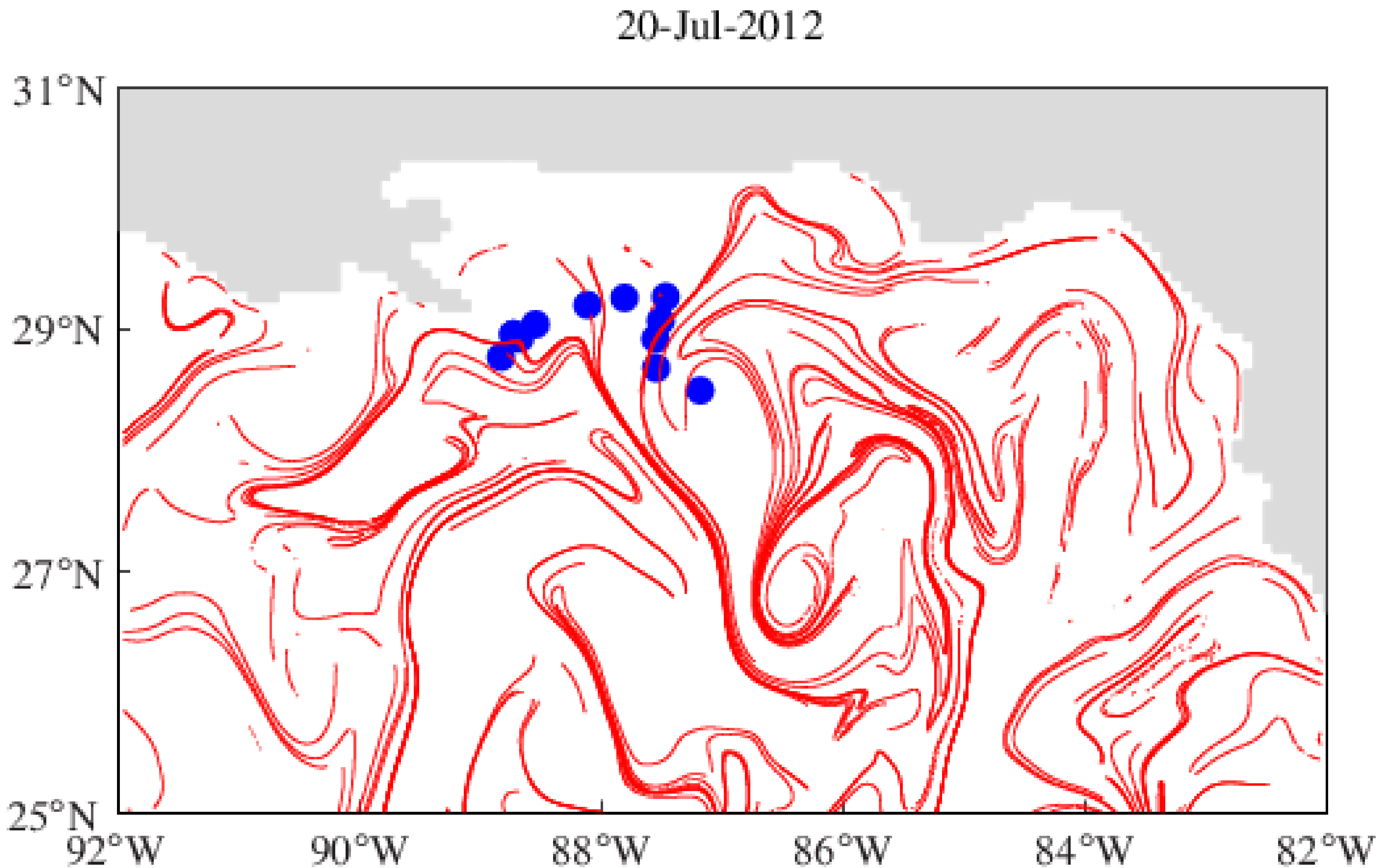
(U, g) : Lorentzian manifold
$r(s)$: has $g = 0$, i.e., null-geodesic

Altimetry-based and simulated LCS vs color



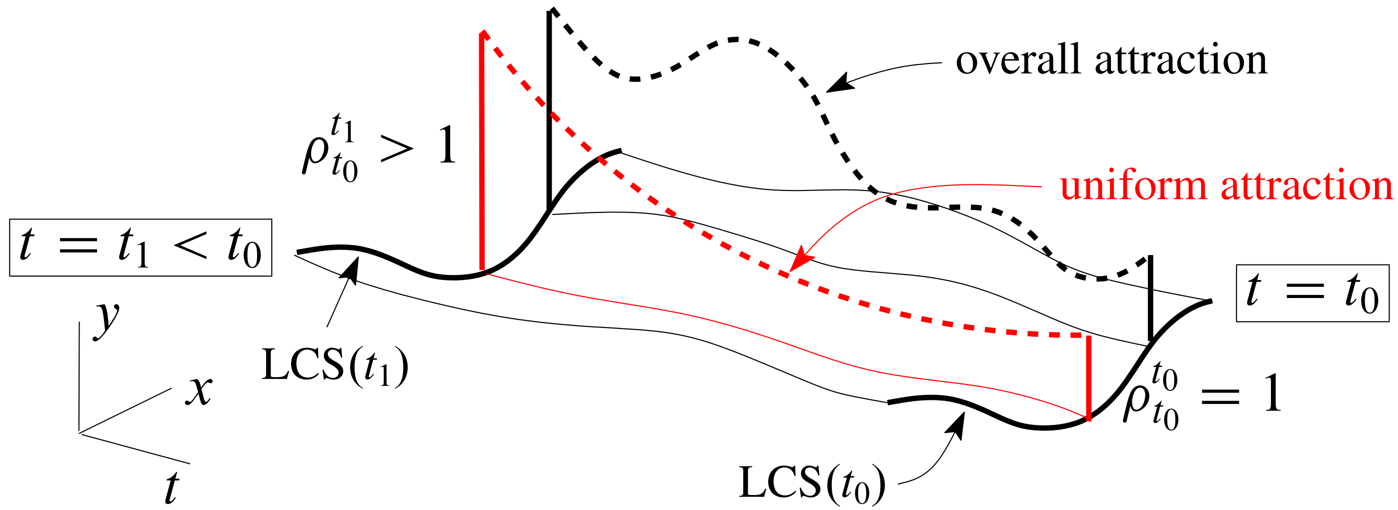
(MJO et al. 2013, preprint)

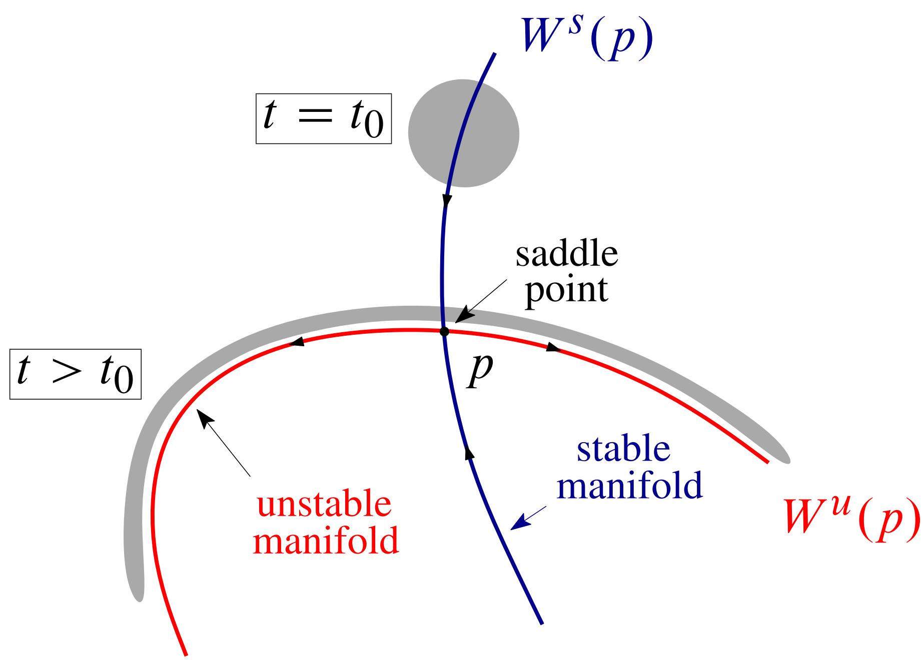
Altimetry-based LCS vs GLAD drifters



(MJO et al. 2013, preprint)

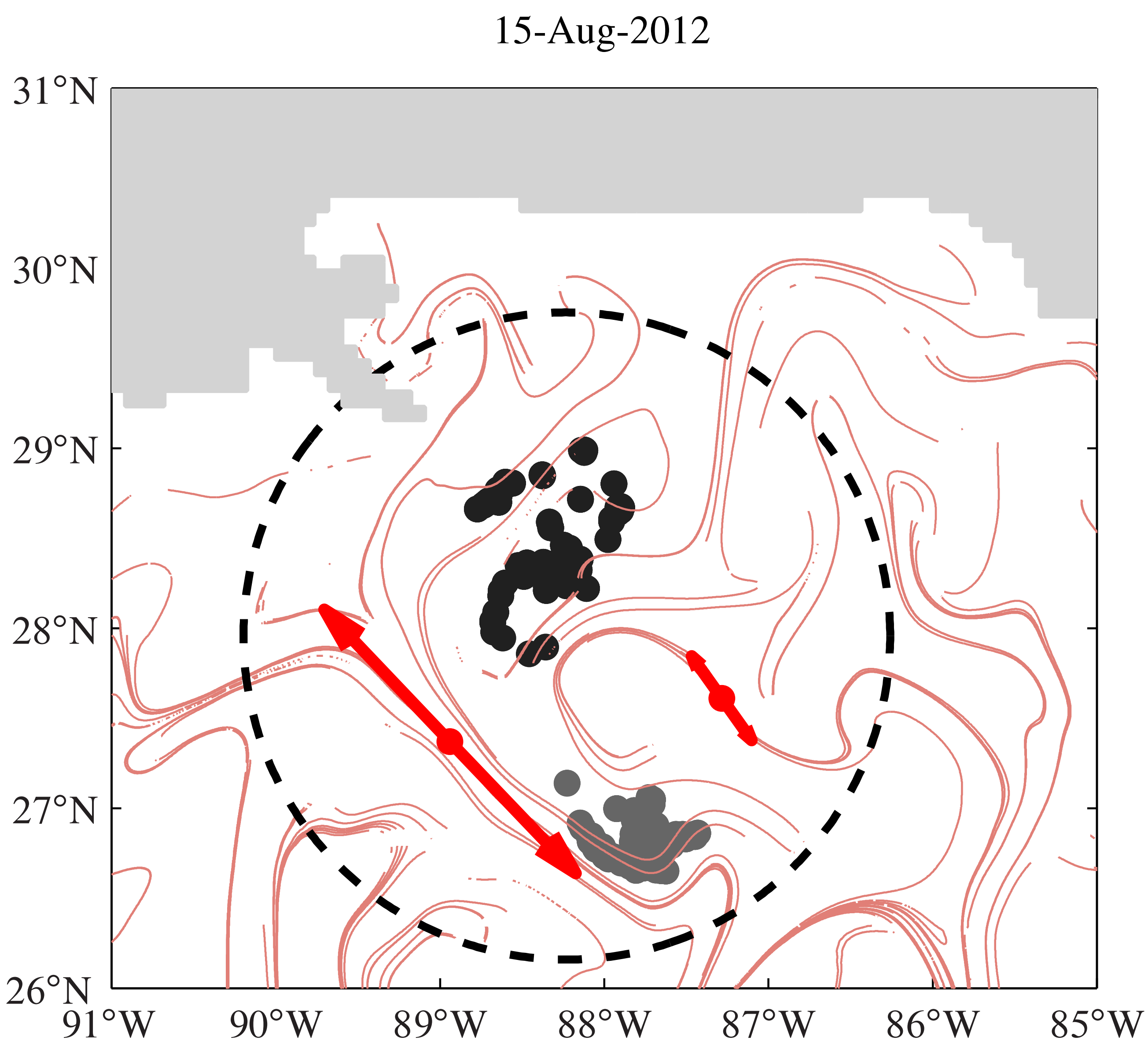
Cores of sustained attraction: Generalized saddles





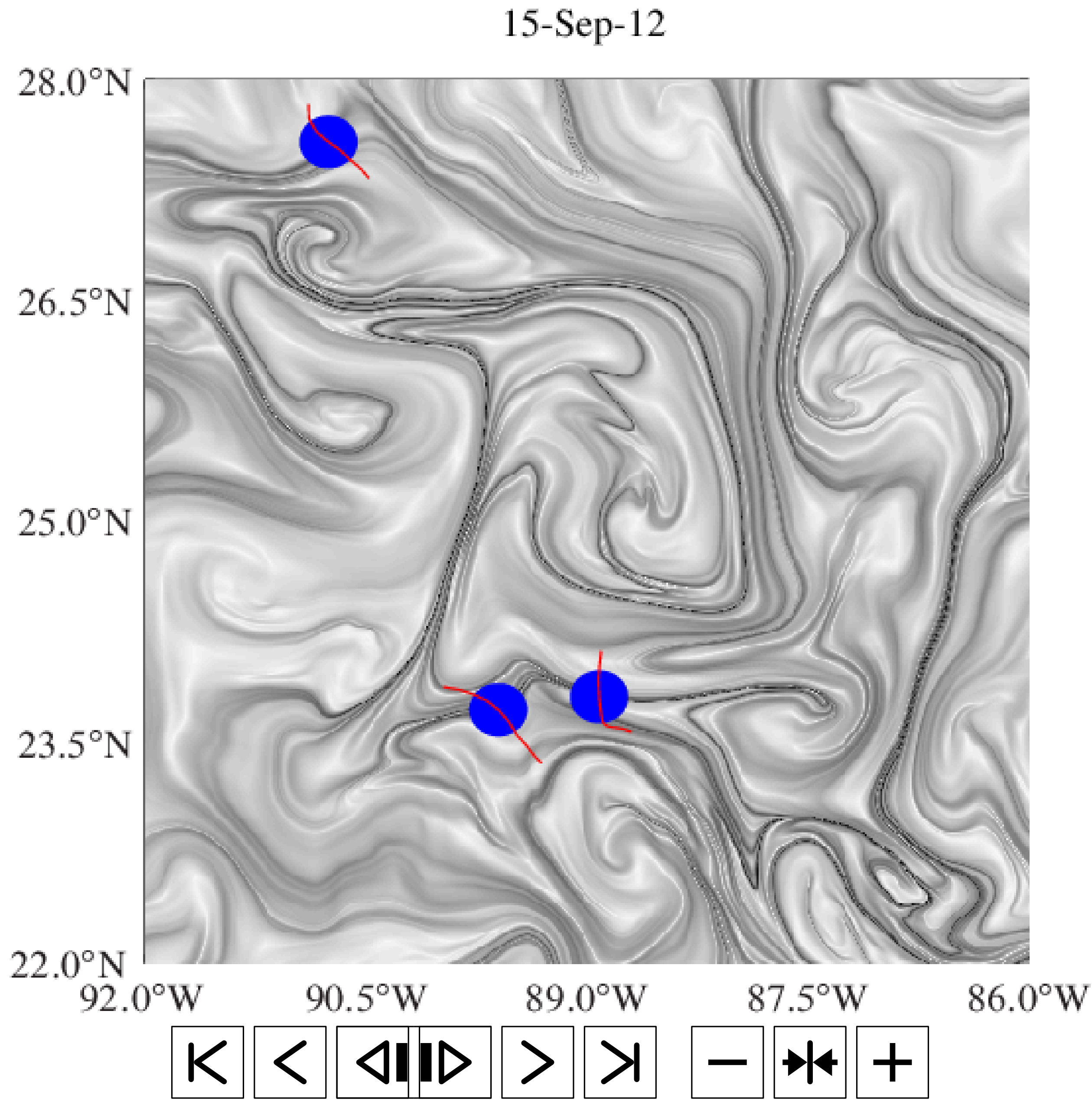
(MJO & GH 2012, *PNAS* 109, 4738)

Cores of sustained attraction during GLAD



(MJO et al. 2013)

Forward stretchlines vs backward FTLE ridges



(FJBV et al. 2013*b*, preprint)

Summary of relevant geodesic LCS types

- **Elliptic LCS**
  - ▷ Outermost $\lambda = 1$ loop (primary BH; super coherent).
  - ▷ Outermost $\lambda \neq 1$ loop (secondary BH; coherent).
- **Hyperbolic LCS**
  - ▷ Least-straining strainlines.
  - ▷ Most-stretching stretchlines.
- **Parabolic LCS**
  - ▷ Strainline/stretchline segments connecting CG singularities.

Thank you.