Geodesic theory of Lagrangian Coherent Structures

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Suggests organizing *Lagrangian Coherent Structure (LCS)*.
Main result

Elliptic LCS (eddy boundaries), hyperbolic LCS (invariant-manifold-like) and parabolic LCS (shear jet cores) as (null-)geodesics of appropriately defined (Lorentzian) metrics.

References

FJBV et al. (2013a). JPO, in press.
**Mathematical setup**

\[ \dot{x} = v(x, t), \quad x \in U \subset \mathbb{R}^2, \quad t \in [t_0, t_1] \subset \mathbb{R} \]

\[ x_t = x(t; x_0, t_0) \]

\[ C_{t_0}^t(x_0) = DF_{t_0}^t(x_0) \top DF_{t_0}^t(x_0) \]

\[ C_{t_0}^t \xi_i = \lambda_i \xi_i, \quad 0 < \lambda_1 \leq \lambda_2, \quad \langle \xi_i, \xi_j \rangle = \delta_{ij} \]

\[ x \mapsto Q(t)x + b(t): \quad C_{t_0}^t \mapsto C_{t_0}^t \]
Why objectivity (frame invariance) matters

\[ v(x, t) = \begin{pmatrix} \sin 4t & 2\cos 4t \\ -2 + \cos 4t & -\sin 4t \end{pmatrix} x \]

\[ \tilde{v}(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{x} \]

Switch to rotating frame:

\[ x = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \tilde{x} \]

Truly unsteady flows have no distinguished frame—remain unsteady in any frame (Lugt, 1979). Conclusions about flow structures should not dependent on frame chosen.
Strainlines, stretchlines, and $\lambda$-lines

$$\mathbb{R}^+ \ni [s_1, s_2] \ni s \mapsto r(s) \in \gamma_0$$

$$r' = \xi_1 \quad \text{or} \quad \xi_2 \quad \text{or} \quad \eta_{\lambda}^{\pm} := \sqrt{\frac{\lambda_2 - \lambda^2}{\lambda_2 - \lambda_1}} \xi_1 \pm \sqrt{\frac{\lambda^2 - \lambda_1}{\lambda_2 - \lambda_1}} \xi_2$$
From strainlines to stretchlines and vice versa

(Farazmand & Haller, 2013)
“The edge of the whirl was represented by a broad belt of gleaming spray; but no particle of this slipped into the mouth of the terrific funnel. . .”
Variational principle

\[ T \ni s \mapsto r(s) \in \gamma_0 : \text{loop} \]

\[
\begin{align*}
l_{t_0}(r') &:= \sqrt{\langle r', r' \rangle}, & l_t(r, r') &:= \sqrt{\langle r', C^t_{t_0}(r)r' \rangle} \\
Q(\gamma_0) &:= \oint \frac{l_t(r, r')}{l_{t_0}(r')} \, ds
\end{align*}
\]

\[ s \mapsto r(s) + \varepsilon h(s) \in \gamma^\varepsilon_0, \quad Q(\gamma^\varepsilon_0) = Q(\gamma_0) + O(\varepsilon^2) \iff \delta Q(\gamma_0) = 0 \]

\[ l^2_t(r, r')/l^2_{t_0}(r') = \lambda^2 = \text{const} \]

\[ \delta \varepsilon_\lambda(\gamma_0) = 0, \quad \varepsilon_\lambda(\gamma_0) := \oint g_\lambda(r)(r', r') \, ds \]

\[ g_\lambda(x_0)(u, u) := \langle u, E_\lambda(x_0)u \rangle \]

\[ E_\lambda(x_0) := \frac{1}{2}(C^t_{t_0}(x_0) - \lambda^2 \text{Id}) \]
Solutions, called \textit{\(\lambda\)-loops}, satisfy implicit ODE:
\[ l_t^2(r, r') - \lambda^2 l_{t_0}^2(r') = \langle r', E_\lambda(r) r' \rangle = 0. \]

Stretch by same factor \(\lambda\) from \(t_0\) to \(t\)—coherent cores of coherent material belts.

We seek \(\lambda\)-loops tangent to linear combinations of eigenvectors of \(C_{t_0}^t\); leads to explicit ODE:
\[ r' = \eta_{\lambda}^\pm(r), \]
\[ \eta_{\lambda}^\pm = \sqrt{\frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1}} \xi_1 \pm \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1}} \xi_2. \]

\(\eta_{\lambda}^\pm\) constitute rotated vector field—\(\lambda\)-loops are nonintersecting (Duff, 1953).

Observable boundary given by outermost of concentric \(\lambda\)-loops.
Cosmological analogy

A each $x_0 \in U_\lambda$ the quadratic form

$$g_\lambda(x_0)(u, u) = \langle u, E_\lambda(x_0)u \rangle$$

defines a Lorentzian metric. Then $(U_\lambda, g_\lambda)$ can be view as Lorentzian 2-manifold—a relativistic spacetime.

Light travels along curves where a Lorentzian metric vanishes—null-geodesics.

Closed null-geodesics (photon spheres) enclose black holes.

Because $\lambda$-loops are closed geodesics of $g_\lambda(r)(r', r') = 0$:

- outermost $\lambda = 1$ loop: *primary black-hole eddy boundary*—super coherent
- outermost $\lambda \neq 1$ loop: *secondary black-hole eddy boundary*—coherent

In the Hamiltonian case ($\text{div} \, \nu = 0$), primary BH eddies preserve area—creates further coherence; most closely related to KAM tori.
Detection of black-hole eddies

- Strain eigenvectors $\xi_1$ and $\xi_2$ become ill-defined at points where $\lambda_1 = \lambda_2 = 1$—singularities of $E_1$.
- Null-geodesics of $g_\lambda$ cannot be extended to such points—$\eta^\pm_\lambda$ are ill-defined there.
- Black holes are believed to contain Penrose–Hawking singularities—analogous to singularities of $E_1$.
- It can be proved (Beem et al., 1999) that close null-geodesics of $g_\lambda$ (i.e., $\lambda$-loops) must contain singularities of $E_1$.
- We exploit this property to detect BH eddy candidate regions.
OCEAN EDDIES & BLACK HOLES

Null-geodesic of the primary Green-Lagrange tensor

Photon sphere of the primary Green-Lagrange tensor

Singularity of the Green-Lagrange metric

Light cone

Aura filled by photon spheres of secondary Green-Lagrange tensors

Ring maximal photon sphere of secondary Green-Lagrange tensors

PRIMARY BLACK-HOLE EDDY
Computation of $\lambda$-loops
BH eddies in the South Atlantic

Metric singularities & black-hole eddy candidates on 24-Nov-06

Black-hole eddy boundaries on 24-Nov-06

Black-hole eddy boundaries on 22-Feb-07
Long-term advection of BH and SSH eddies

TRANSPORT OF WATER BY BLACK-HOLE EDDIES

TRANSPORT OF WATER BY A SEA-SURFACE-HEIGHT EDDY

Transported water after 0, 45, 90, 135, 180, and 225 days
Robustness under velocity degradation

Direct consequence of structural stability of limit cycles.
Reality check: surface ocean chlorophyll

24/11/06

22/02/07

23/05/07
Shear vanishes locally.
Variational principle (Farazmand et al. 2013)

\[
\sigma (r, r') = \frac{\langle r', D(r)r' \rangle}{\sqrt{\langle r', C_{t0}^t (r)r' \rangle \langle r', r' \rangle}}
\]

\[
D := \frac{1}{2} (C_{t0}^t \Omega - \Omega C_{t0}^t)
\]

\[
\Sigma (\gamma_0) := \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \sigma (r, r') \, ds
\]

\[
\Sigma (\gamma_0^\varepsilon) = \Sigma (\gamma_0) + O(\varepsilon^2) \iff \delta \Sigma (\gamma_0) = 0
\]

\[
\delta \Sigma (\gamma_0) = \langle \partial_{r'} \sigma, h \rangle|_{s_1}^{s_2} + \int_{s_1}^{s_2} \left( \partial_r \sigma - \frac{d}{ds} \partial_{r'} \sigma \right) h \, ds
\]

free : \( C_{t0}^{t_1} (r(s_1)) = C_{t0}^{t_1} (r(s_2)) = \text{Id} \)

fixed : \( h(s_1) = h(s_2) = 0 \)
Variational principle (Farazmand et al. 2013)

\[
\frac{\langle r', D(r)r' \rangle}{\sqrt{\langle r', C_{t_0}^t(r)r' \rangle \langle r', r' \rangle}} = \mu = \text{const}
\]

\[
\mu = 0 : \langle r', D(r)r' \rangle = 0 \iff r' \parallel \xi_i
\]

**parabolic LCS** : strain- and stretchlines connecting CG singularities

**hyperbolic LCS** : strain- or stretchlines

\[
\delta \mathcal{E}(\gamma_0) = 0, \quad \mathcal{E}(\gamma_0) := \int_{s_1}^{s_2} g(r)(r', r') \, ds
\]

\[
g(x_0)(u, u) := \langle u, D(x_0)u \rangle
\]

\[(U, g) : \text{Lorentzian manifold}
\]

\[r(s) : \text{has } g = 0, \text{ i.e., null-geodesic}\]
Altimetry-based and simulated LCS vs color

(MJO et al. 2013, preprint)
Altimetry-based LCS vs GLAD drifters

(MJO et al. 2013, preprint)
Cores of sustained attraction: Generalized saddles

\[ \rho^{t_1}_{t_0} > 1 \]

\[ \rho^{t_0}_{t_0} = 1 \]

\[ t = t_1 < t_0 \]

\[ t = t_0 \]

\[ t > t_0 \]

\( W^s(p) \)

\( W^u(p) \)

(MJO & GH 2012, PNAS 109, 4738)
Cores of sustained attraction during GLAD

(MJO et al. 2013)
Forward stretchlines vs backward FTLE ridges

(FJBV et al. 2013b, preprint)
Summary of relevant geodesic LCS types

● Elliptic LCS
  ▶ Outermost $\lambda = 1$ loop (primary BH; super coherent).
  ▶ Outermost $\lambda \neq 1$ loop (secondary BH; coherent).

● Hyperbolic LCS
  ▶ Least-straining strainlines.
  ▶ Most-stretching stretchlines.

● Parabolic LCS
  ▶ Strainline/stretchline segments connecting CG singularities.
Thank you.